Scalable Methods for Incompressible Stokes Flow with Yielding Rheologies

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Abstract: We investigate the techniques needed to make a stable, robust method for solving large, 3D, highly nonlinear lithospheric problems.

Instability of Q1-P0 elements

Many finite element codes developed for crustal scale problems use Q1-P0 elements. This scheme has a well known checkerboard instability. Figure 1 shows a rather mild example.

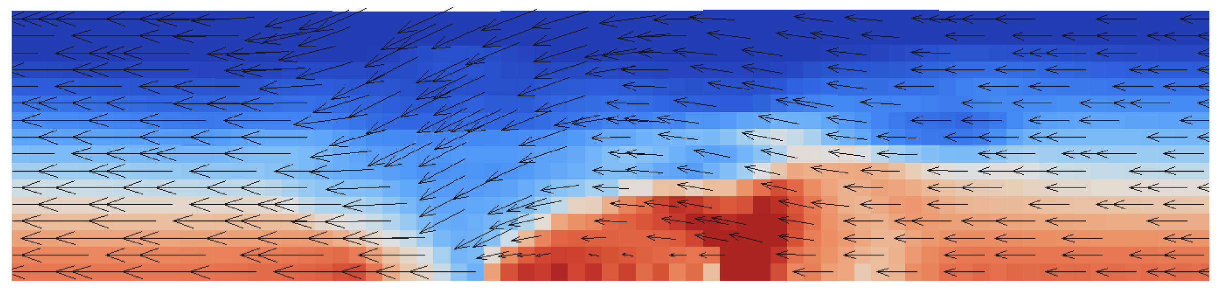

Figure 1: Pressure for a subduction model using Q1-P0 elements and averaged pressures. Notice that, even when averaging the pressures, there is an even-odd artifact centered on the bottom.

When the problem becomes strongly non-linear, such as with yielding rheologies, this instability can cause the method to break down.

There have been a number of fixes including higher order elements, macro-elements, and higher order derivatives. They are relatively complex and difficult to integrate with particle based methods for tracking properties of particles like accumulated strain, needed for yielding rheologies.

Simple Q1-Q1 Fix

Dohrmann and Bochev (2000) introduced a relatively simple scheme using Q1-Q1 elements and a stabilization term that looks like a compressibility. Specifically, the continuity equation is modified to become

$$\nabla \cdot \mathbf{v} + Cp = 0$$

where

$$C \equiv \frac{1}{\nu}\left(\int_{\Omega_e} \psi(x)\psi^T(x)d\Omega_e - \int_{\Omega_e} d\Omega_e\right)$$

$\mathbf{v}$ is the velocity, p is the pressure, ν is the viscosity, Ω_e is the element and $\psi(x)$ is the basis function. Thus the compressiblity is scaled by the viscosity and is proportional to the mass matrix and the element area. So as the resolution increases, the compressibility converges to zero, giving us the original incompressible flow. For problems the size of a sandbox, this method works quite well. Figure 2 show a sandbox model with narrow shear zones and high viscosity contrasts.

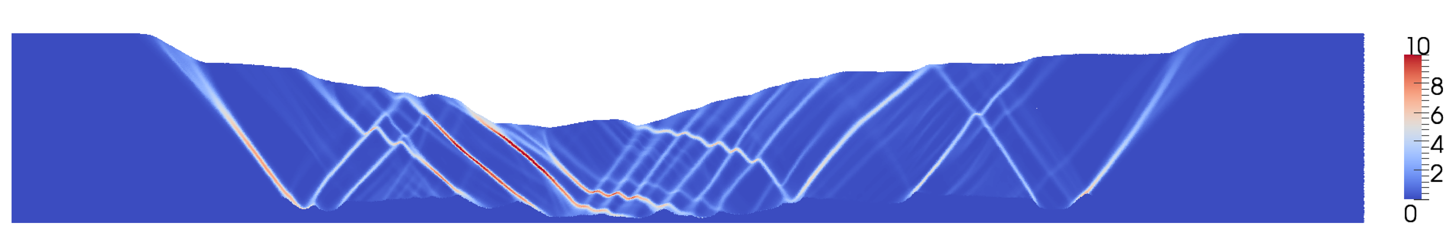

Figure 2: Accumulated strain for the 2004 Geomod extension benchmark

The Fix Causes Excess Compressibility

Unfortunately, when the method is applied to the lithosphere, serious problems arise. In our sandbox example, the pressure is dominated by the dynamic pressure, not the hydrostatic pressure. The dynamic pressure, in turn, is created by the motions arising from the velocities. Because of this link between the velocity and the pressure, the overall magnitude of the extra Cp term tends to be relatively small.

In lithospheric problems, on the other hand, the hydrostatic pressure can be several orders of magnitude larger than the dynamic pressure. Since the hydrostatic pressure is not linked to the velocity, it is easy for the extra term Cp to be far too large, even though C itself is small and gets smaller as resolution increases. Figure 3 shows how the dynamics of a simple extension model can get completely disrupted.

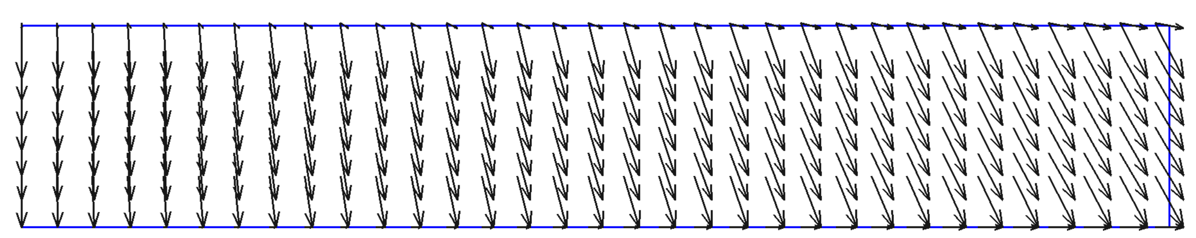

Figure 3: A simple box of viscous material being pulled on the right side while the left is held fixed. The motion is primarily straight down as the material collapses on itself because of spurious compressibility.

Subtract the Problem Away

An obvious way to fix this is to subtract out the hydrostatic part of the pressure. We do this by subtracting out a nominal density profile. This removes the hydrostatic pressure associated with the nominal density profile. When computing the true pressure (e.g., for yielding calculations or visualization) we have to remember to add it back in.

As will be shown later, it is important to make a decent choice for the density profile, although it need not be perfect. For simple geometries, this is relatively easy to do and results in a better solution, as shown in Figure 4.

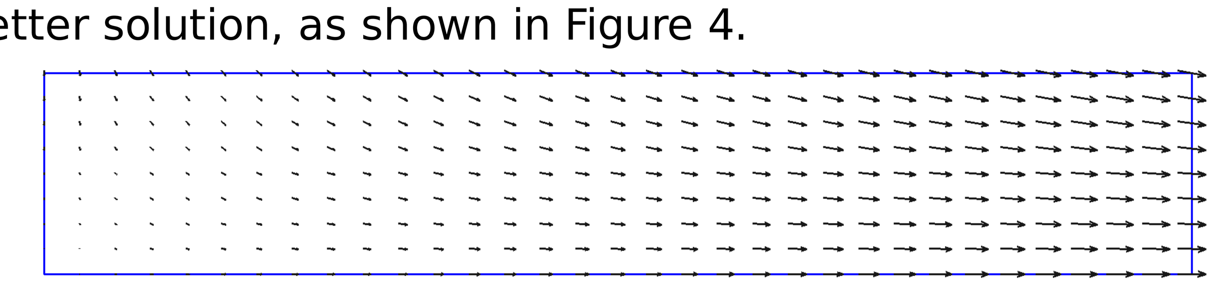

Figure 4: The same setup as in Figure 3, but with a nominal density profile subtracted away. The spurious compressibility has all but disappeared.

Boundary Conditions

A side effect of this scheme is that boundary conditions become more complicated. For example, consider a simulation where the bottom boundary is open, allowing material to freely flow in and out.

Normally, we could apply a stress at the bottom to counteract the weight of the material. Then, as shown in Figure 5, when material piles up, the weight of the material will overcome the stress boundary, and material will flow out. Conversely, when material thins out, the stress boundary will push new material into the simulation.

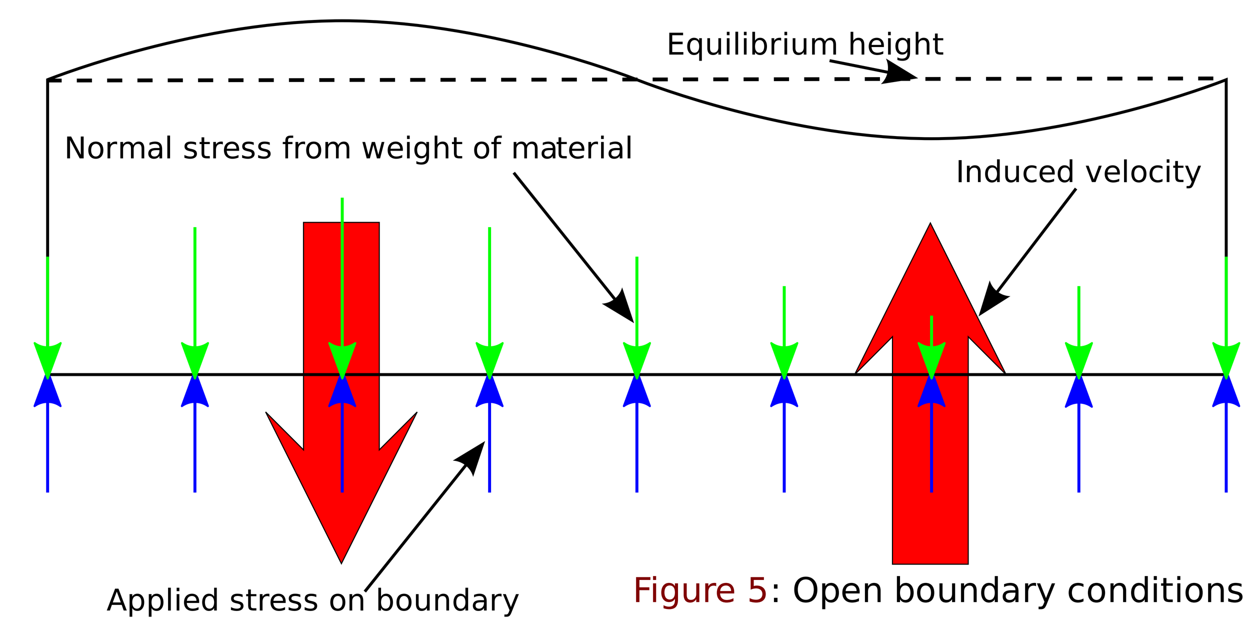

Figure 5: Open boundary conditions

However, the nominal density profile has been subtracted away, so there is no need to counteract the hydrostatic pressure. If material piles up, then the excess will not be subtracted away. That excess material then exerts a force that pushes the material down and material flows out as expected.

The converse is not true, though. If the material thins out, then there is no force on the bottom pushing the material up. We are still subtracting the nominal profile, so there should be a negative pressure resulting from the negative density above the simulation. To model this, we apply a stress to the top of the domain that is equal to the nominal pressure from the nominal density. As shown in Figure 6, that pulls up the material when it sinks too low.

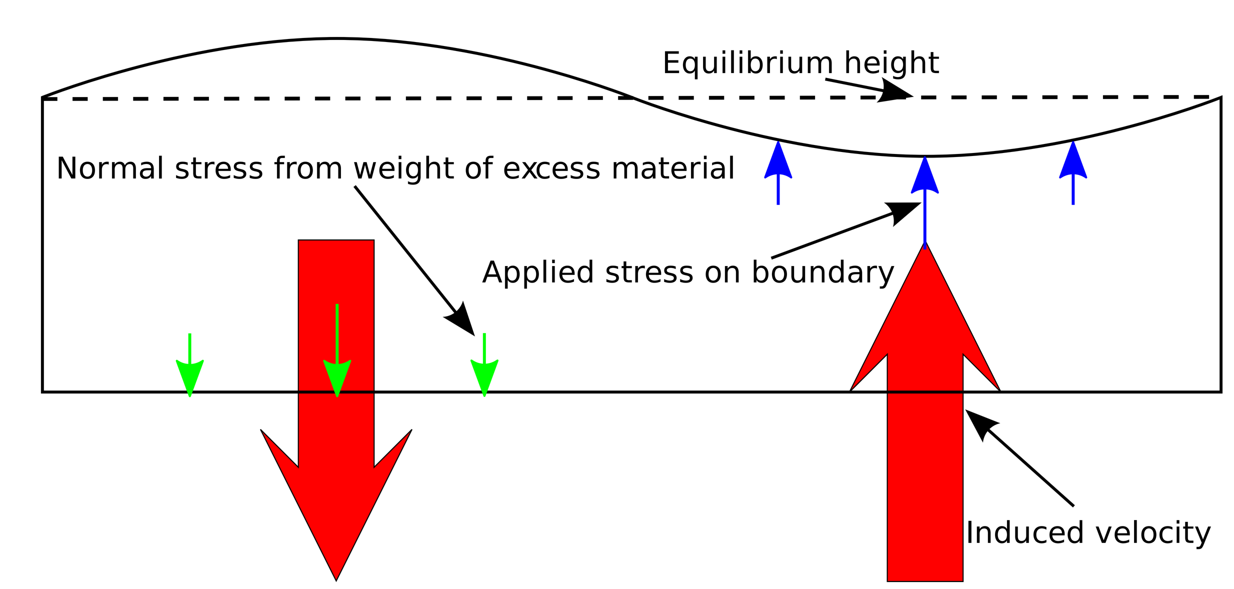

Figure 6: Open boundary conditions with a nominal density profile

Dynamics Reveal an Instability

With complex geometries, it is not always clear how to compute a nominal density profile. For example, Figure 7 shows a simulation of a tabletop extension experiment over time. The side is pulled slowly enough such that hydrostatic pressure is much larger than dynamic pressure. The material is suspended in an asthenospheric fluid that flows up to fill the void as the mantle and crust thin. The weak zone serves as a seed for yielding. The asthenosphere is more dense than the mantle, which is more dense than the crust and weak zone.

A reasonable scheme for subtracting out the hydrostatic pressure is to subtract the profile of the asthenosphere, mantle, and crust. We can not vary the profile in space, because that would cause dynamics that are then not accounted for. So the weak zone will not be exactly subtracted, but as the simulation shows, this does not seem to cause any problems.

As the crust thins, the heavy asthenospheric material flows up to fill in the void left by the weak zone. However, this triggers an interesting instability. The heavier material creates an excess pressure. This pressure makes the Cp term larger. There is then a coupling which causes even more pressure, which increases Cp, and so on.

The fix for this is to change the nominal density profile so that it is all made up of the heavier material. The mechanism behind the instability and the workaround is not completely clear.

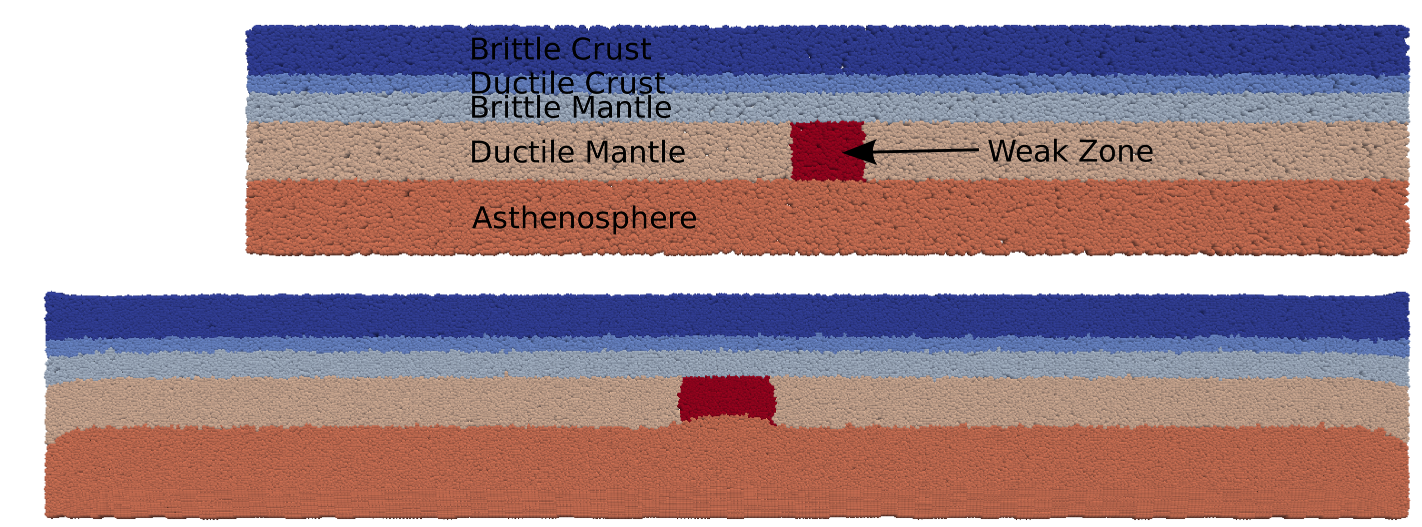

Figure 8: Simulation of a tabletop model with multiple layers. As the material extends, the weak zone thins more quickly, allowing the dense asthenosphere to fill the void. If the nominal density profile is set up to reflect the differing densities in the crust, mantle, and asthenosphere, then an instability occurs where the asthenosphere is filling in the void. If, instead, the profile assumes that the dense asthenosphere is the only material, then the simulation succeeds. Model courtesy of John Sheehan.

Conclusion

The techniques presented here make it possible to simulate highly nonlinear Stokes flow problems with the relatively simple Q1-Q1 finite elements. This allows us to use particle methods in a straightforward manner. That, in turn, lets us capture strain history, allowing us to follow simulations with large deformations with no diffusion.

With that said, the introduction of the nominal density profile adds as many problems as it solves. Getting all of the parameters just right can be an arduous exercise.

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