Modeling and Testing Yielding and Magmatic Dikes

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For numerical models of the lithosphere, there are a number of processes that can be quite crucial to understanding the underlying phenomena. We focus on how to model and, more importantly, test two key features in the context of Gale, a parallel 2D/3D Stokes flow code.

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Mohr-Coulomb Yielding

Theory: For the Mohr-Coulomb rheology, we can write the yielding relation as

$$\sigma_{ns} = \tan\varphi\, \sigma_{nn} + C$$

where σ_{ns} is the shear stress perpendicular to the fault plane, σ_{nn} is the stress normal to the plane, φ is the angle of internal friction, and C is cohesion. Decomposing this into principal stresses gives us

$$\sin(2\Theta)(\sigma_I - \sigma_{III})/2 = \tan\varphi((\sigma_I + \sigma_{III})/2 + \cos(2\Theta)(\sigma_I - \sigma_{III})/2) + C$$

where Θ is the angle that the fault makes relative to the maximum shear stress. We then assume that the fault forms when the shear stress $(\sigma_I - \sigma_{III})/2$ is a minimum. After a little algebra, this gives

$$\Theta = +/-(\pi/4 + \varphi/2)$$

Numerical Shortening Experiment: We performed a simple shortening experiment as in Figure 1 for a variety of φ. We solve the Stokes equation and look at the strain rate invariant to find incipient faults. This removes any of the confounding effects of strain weakening or time stepping.

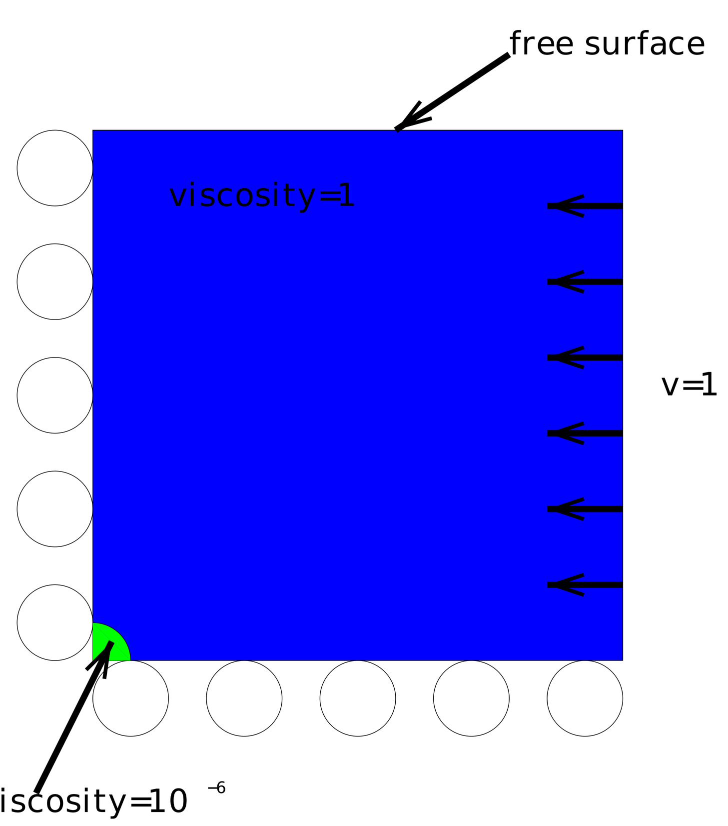

Figure 1: The set up for the shortening experiment. The box is 1 unit long on a side, and the low viscosity region has a radius of 0.01 (its size is exaggerated).

Figure 2 shows a result for three different resolutions for φ=45 degrees. Figure 3 shows a plot of the calculated vs expected values of the fault angles. This agreement gives us confidence in our implementation of Mohr Coulomb.

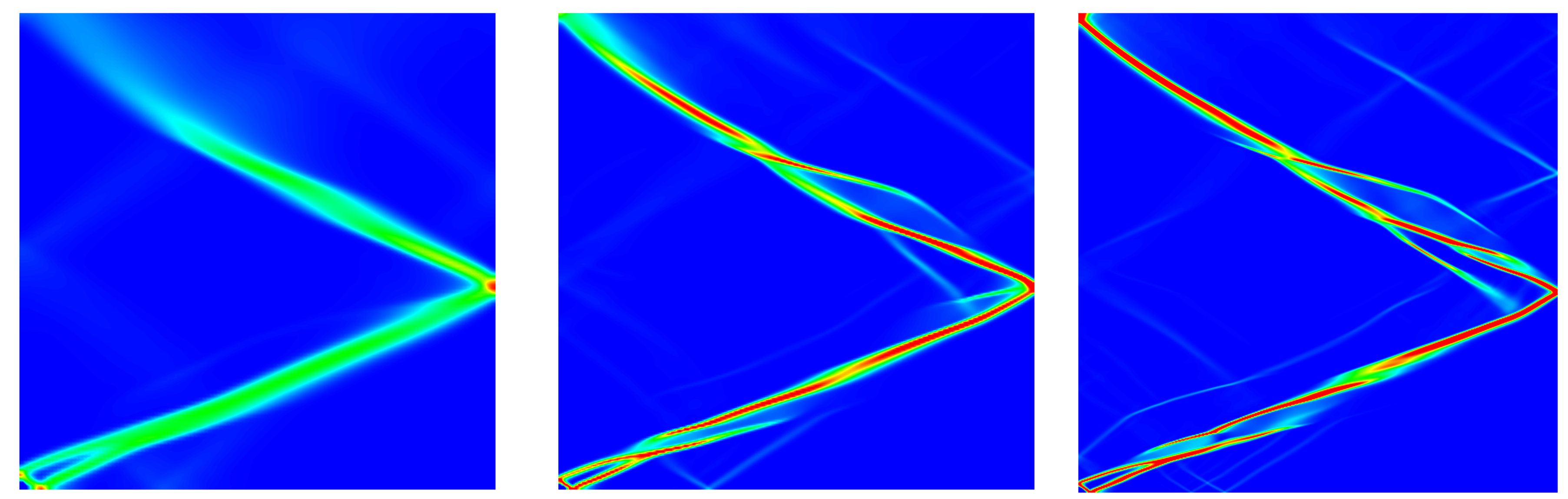

Figure 2: Strain rate invariant for the shortening experiment for resolutions of 128x128, 256x256, and 512x512. While differences continue to crop up as resolution is increased, there is good convergence towards a specific fault angle.

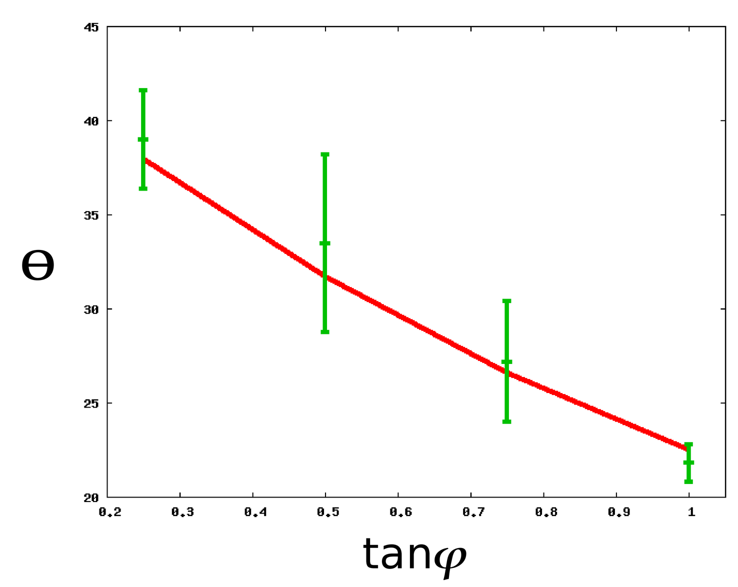

Figure 3: Analytic versus numerical results for fault angles as a function of friction angle.

Numerical Extension Experiment: We also performed a simple extension experiment where, instead of pushing the side of the box to the left, we pulled it to the right. This should be almost the same as switching the top and right side of the shortening experiment. Figure 4 shows a typical result, regardless of resolution and φ. The angle is always 45 degrees, as if the φ were always zero. This arises from numerical necessities. Pulling in this manner can give rise to large negative pressures in the intermediate solutions. These negative pressures would then cause spurious yielding in places with very low stresses throughout the domain. To control this, so we set a minimum yield stress. So in regions with negative pressures, we are essentially using a Von Mises rheology.

Figure 4: Strain rate invariant for the extension experiment. While this model has a resolution of 64x64 and $\tan\varphi$=0.5, this result is typical for all resolutions and φ .

Magmatic Dikes

Theory: In the context of Stokes flow simulations, a useful approximation is to treat magmatic dikes as a region where material is created from thin air and flows out. We start by writing down the Stokes equations

$$\sigma_{ij,j} + p_{,i} = f_i$$
$$\nabla \cdot u = 0,$$

where

σ=stress

p=pressure

f =body forces (e.g. gravity)

u=velocity.

We can introduce creation or destruction of material by adding a divergence term to the continuity equation

$$\nabla \cdot u = D.$$

The question is then: how does this divergence term affect the dynamics of the simulation? To clarify this, consider an isoviscous case. In that case, the stress becomes

$$\sigma_{ij} = \boldsymbol{\eta}(u_{i,j} + u_{j,i}),$$

where

$\boldsymbol{\eta}$=viscosity.

Putting in the divergence term

$$\boldsymbol{\eta}(u_{i,jj}) + D_{,i} + p_{,i} = f_{,i}$$

In the magma chamber, the viscosity is low, so we approximate cross derivatives as zero. Restricting temporarily to one dimension,

$$u_{x,x} = D$$
$$\boldsymbol{\eta}(u_{x,xx} + D_{,x}) + p_{,x} = f_{,x}$$
$$\Rightarrow \boldsymbol{\eta} D_{,x} + p_{,x} = f_{,x}$$

Subtracting out the hydrostatic equilibrium implies

$$p = -2\boldsymbol{\eta} D + \text{constant}$$

So if there is a constant divergence inside the magma chamber and zero outside, this creates a pressure jump of $2\boldsymbol{\eta}D$.

You can also go through the calculations for a sphere or ellipsoid and get the same result.

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