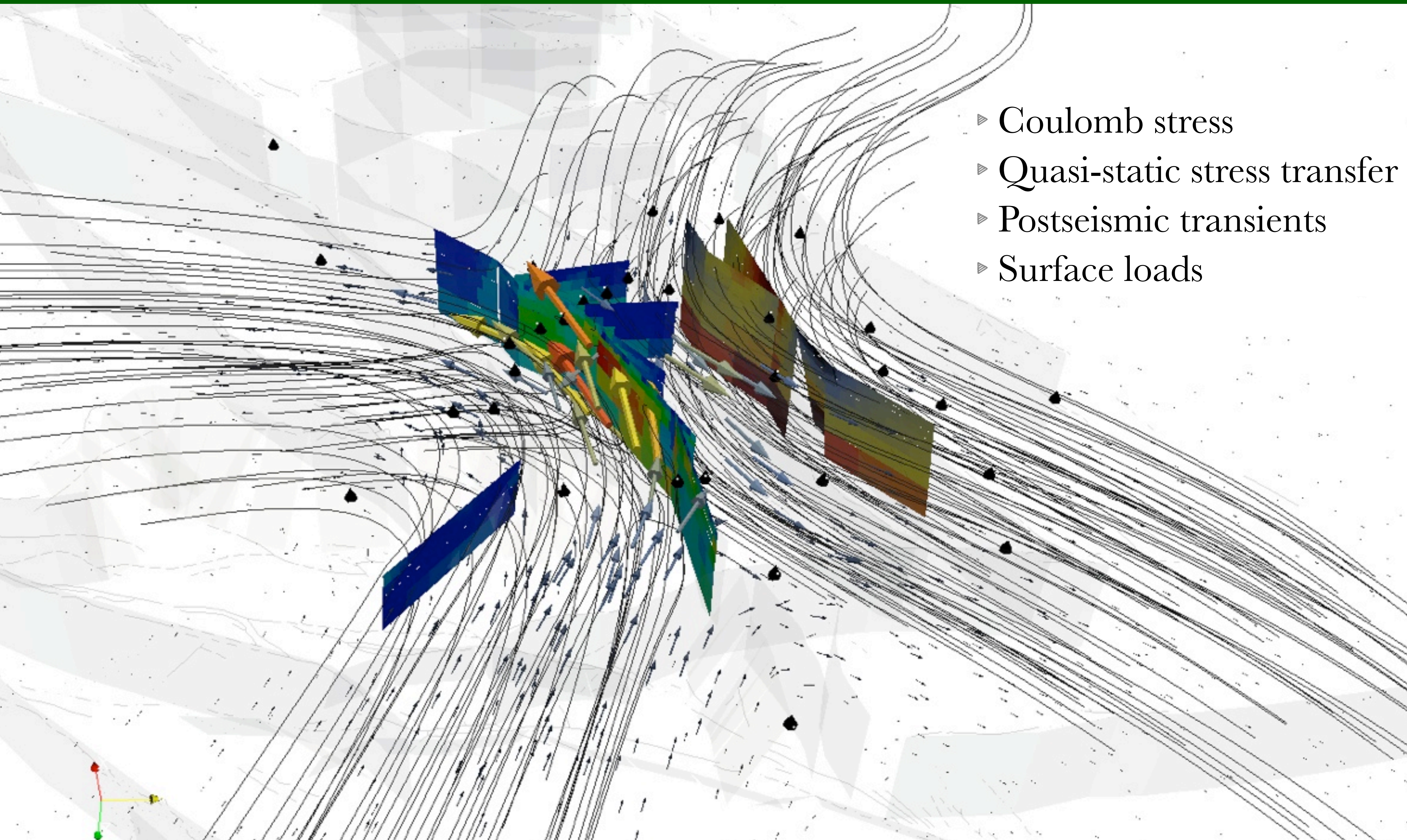
Relax

Semi-analytic Fourier-domain solver and equivalent body forces for quasi-static relaxation of stress perturbation

Sylvain Barbot
California Institute of Technology

Relax



- Coulomb stress
- Quasi-static stress transfer
- Postseismic transients
- Surface loads

Relax - Outlines

- Equivalent body forces for faulting
- Elastic Green's functions in the Fourier domain
  - Examples
- Equivalent body force for viscoelastoplastic problems
  - Examples

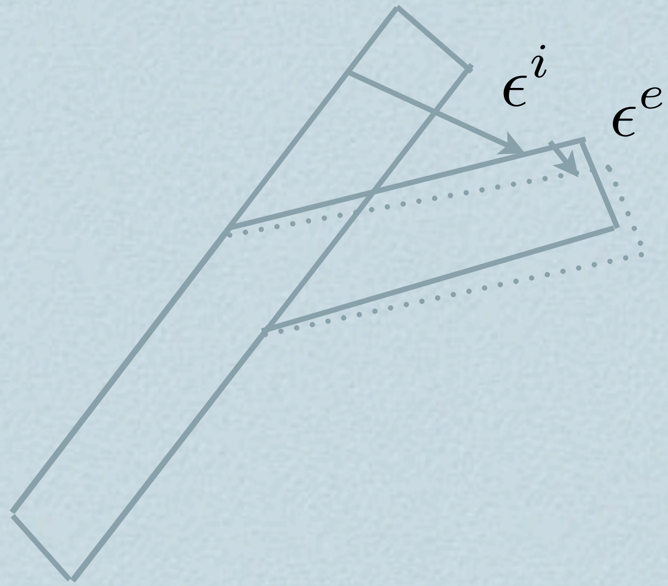
Equivalent body forces

Total strain is decomposed in elastic and inelastic strain components

$$\epsilon_{ij} = \epsilon_{ij}^e + \epsilon_{ij}^i$$

Stress is the result of elastic (reversible) strain

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl}^e$$



Conservation of momentum (Newton's law)

in the interior ⟶ $\sigma_{ij,j} = 0$

$\sigma_{ij}\hat{n}_j = 0$ ⟵ at the surface

inhomogeneous governing (Navier's) equation:

$$\left(C_{ijkl}\epsilon_{kl}\right)_{,j} + f_i = 0$$

equivalent body force (source term):

$$f_i = -\left(C_{ijkl}\epsilon_{kl}^i\right)_{,j}$$

boundary condition:

$$\left(C_{ijkl}\epsilon_{kl}\right)\hat{n}_j + t_i = 0$$

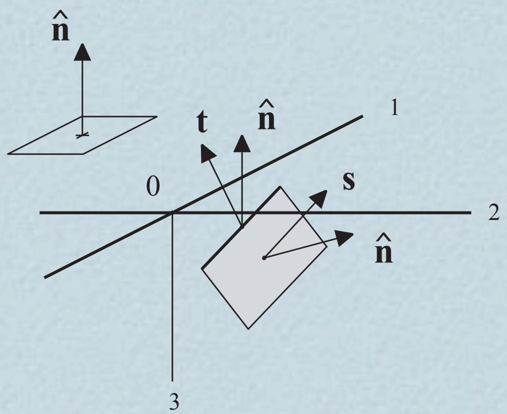
equivalent surface traction:

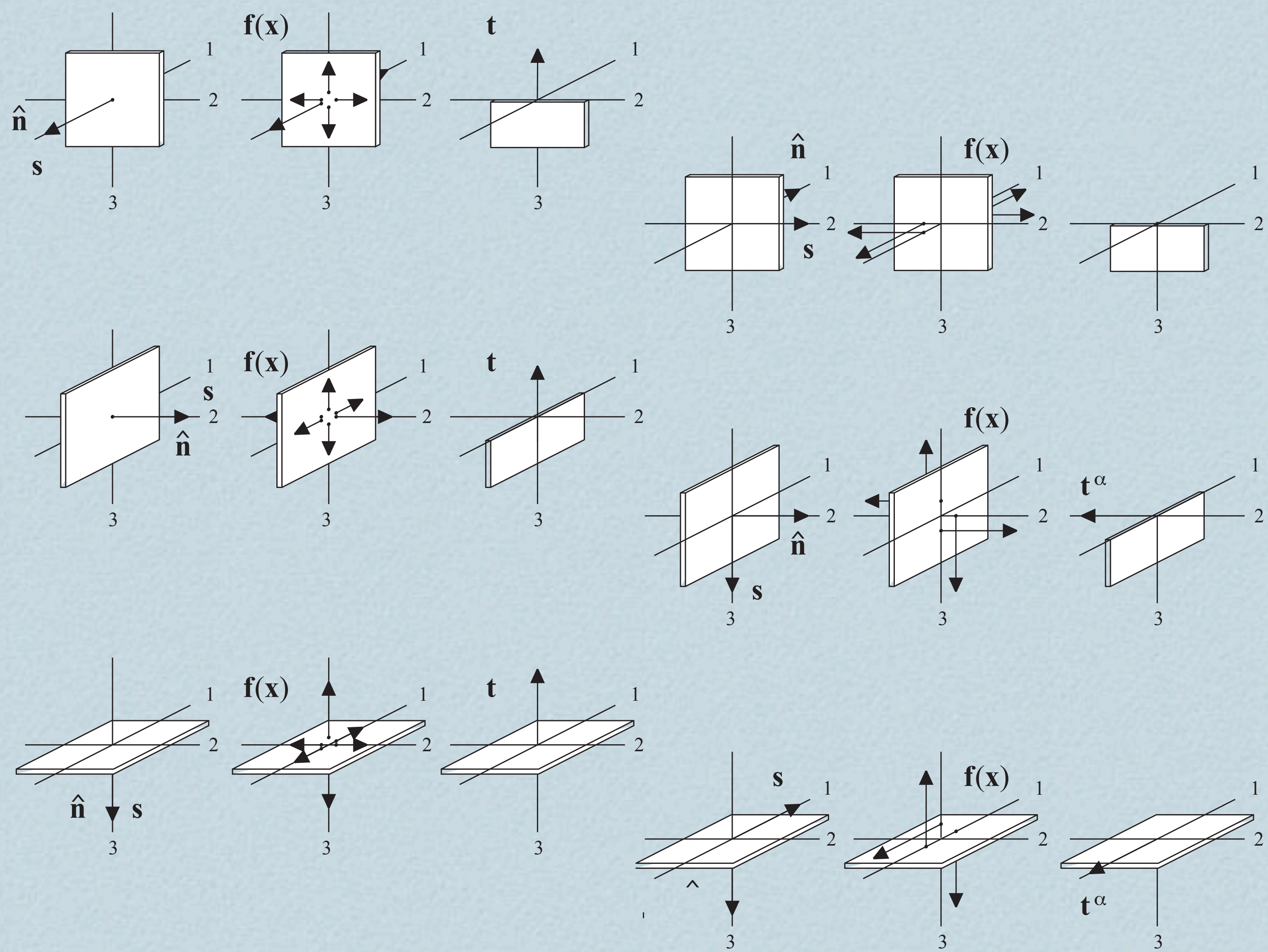
$$t_i = -C_{ijkl}\epsilon_{kl}^i\hat{n}_j$$

Equivalent body forces - Faulting

For a dislocation, the eigenstrain depends on the slip direction, the fault orientation and dimension.

$$\epsilon^i_{ij} = \frac{1}{2}\left(s_i\hat{n}_j + \hat{n}_j s_i\right)\Omega(x_i)$$





equivalent body force:

$$f_i = -\left(C_{ijkl}\epsilon^i_{kl}\right)_{,j}$$

equivalent surface traction:

$$t_i = -C_{ijkl}\epsilon^i_{kl}\hat{n}_j$$

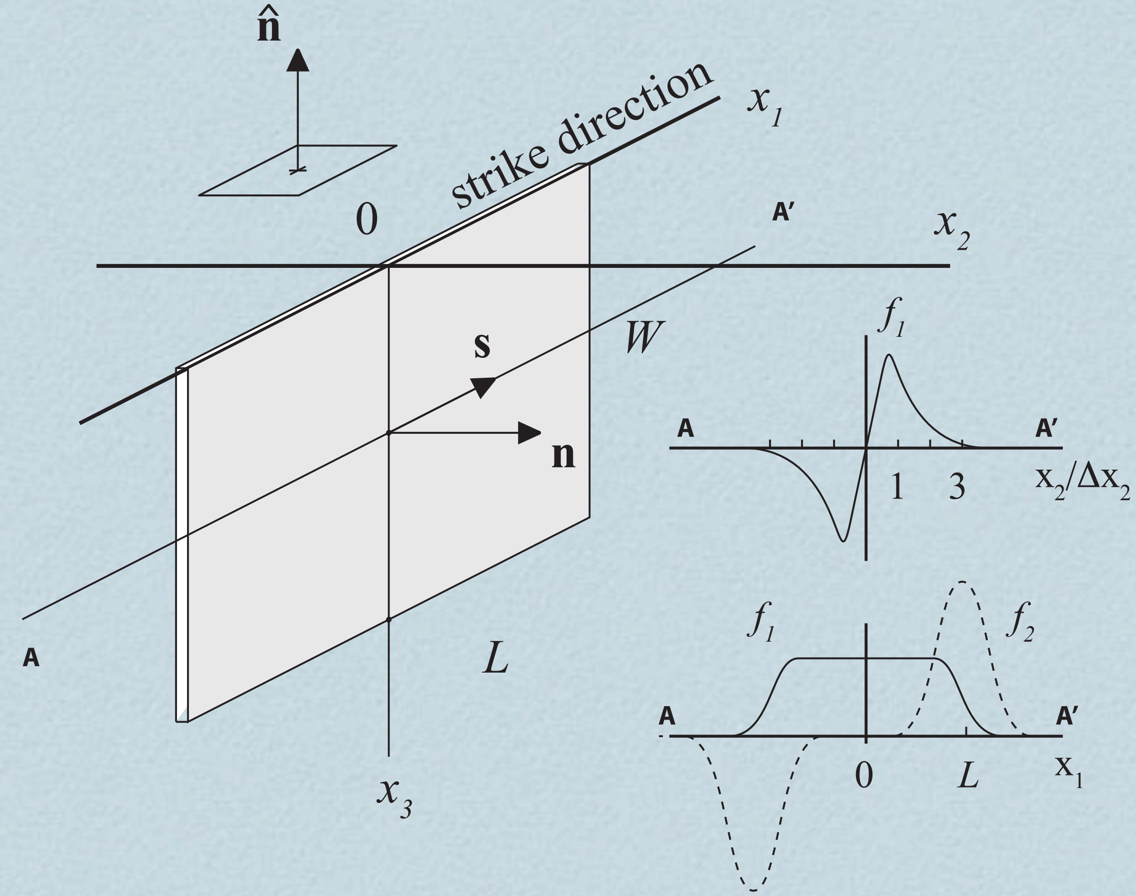
Equivalent body forces - Faulting

For the special case of a vertical left-lateral strike-slip fault, the equivalent body force field is:

$$f_1(x_1, x_2, x_3) = -G\,\Omega_\beta\left(\frac{x_1}{L}\right)\frac{\partial}{\partial x_2}\delta_T(x_2)\,\Omega_\beta\left(\frac{x_3}{W}\right)$$

$$f_2(x_1, x_2, x_3) = -G\,\frac{\partial}{\partial x_1}\Omega_\beta\left(\frac{x_1}{L}\right)\delta_T(x_2)\,\Omega_\beta\left(\frac{x_3}{W}\right)$$

$$f_3(x_1, x_2, x_3) = 0$$



thick faults

For numerical considerations, the Delta and Heaviside functions are tapered:

$$\delta_T(x) = \frac{1}{T\sqrt{2\pi}}\exp\left(-\frac{x^2}{2T^2}\right)$$

$$\Omega_\beta(x) = \begin{cases} 1, & |x| < \dfrac{1-2\beta}{2(1-\beta)} \\ \cos\left(\pi\dfrac{(1-\beta)|x| - \frac{1}{2} + \beta}{2\beta}\right)^2, & \dfrac{1-2\beta}{2(1-\beta)} < |x| < \dfrac{1}{2(1-\beta)} \\ 0, & \text{otherwise} \end{cases}$$

Elastic Green's function

Given equivalent body forces and surface tractions for faulting, displacement is obtained with elastic Green's functions:

$$\mathbf{u}_0(\mathbf{x}) = \int_\Omega \mathbf{G}(\mathbf{x},\mathbf{x}_0)\cdot\mathbf{f}(\mathbf{x}_0)\; d\mathbf{x}_0 + \int_{\partial\Omega} \mathbf{G}(\mathbf{x},\mathbf{x}_0)\cdot\mathbf{t}(\mathbf{x}_0)\, dA$$

For example, for a full space, the elastic Green's function can be expressed as follows

$$G_{ij} = \frac{1}{16\pi G(1-\nu)}\left[\frac{(3-4\nu)\delta_{ij}}{R} + \frac{r_i r_j}{R^3}\right]$$

Computing displacements at N locations due to a distribution of forces requires a computational burden scaling with N^2. Computing the elastic response numerically **in the Fourier domain is more efficient, scaling with N log N**.

Fourier-domain Green's function

For efficient calculations, we solve **Navier's equation in the Fourier domain**.

governing equation:

$$\mu \left(\frac{\alpha}{1-\alpha} u_{j,ij} + u_{i,jj} \right) + f_i = 0$$

boundary condition:

$$\sigma_{ij} \hat{n}_j = q_j$$

The solution to the inhomogeneous equation can be decomposed into two terms,

$$u_i = u_i^h + u_i^p$$

with the homogeneous

$$\alpha\, u^h_{j,ij} + (1-\alpha)\, u^h_{i,jj} = 0$$

and inhomogeneous components

$$\mu \left(\frac{\alpha}{1-\alpha} u^p_{j,ij} + u^p_{i,jj} \right) + f_i = 0$$

Barbot & Fialko (2010a)

Fourier-domain Green's function

Inhomogeneous term:

The inhomogeneous term does not necessarily satisfy the boundary condition but, in the interior, obeys

$$\mu\left(\frac{\alpha}{1-\alpha}u^p_{j,ij}+u^p_{i,jj}\right)+f_i=0$$

Upon Fourier transforming, the governing equation becomes algebraic

$$\mu\left(\frac{\alpha}{1-\alpha}k_ik_j+k_lk_l\,\delta_{ij}\right)\hat{u}^p_j=\frac{1}{4\pi^2}\hat{f}_i$$

and the Fourier-domain solution is simply:

displacement

$$\boxed{\hat{u}^p_i=\frac{1}{\mu}\frac{(1-\alpha)\,k_lk_l\,\delta_{ij}-\alpha\,k_ik_j}{4\pi^2(k_lk_l)^2}\,\hat{f}_j}$$

force

transfer function

Fourier-domain Green's function

Homogeneous term:

The homogeneous term satisfies

$$\alpha\, u^h_{j,ij} + (1-\alpha)\, u^h_{i,jj} = 0$$

and the boundary condition:

$$\sigma_{13}(x_1,x_2) = p_1(x_1,x_2)$$
$$\sigma_{23}(x_1,x_2) = p_2(x_1,x_2)$$
$$\sigma_{33}(x_1,x_2) = p_3(x_1,x_2) - \Delta\rho\, g\, u^h_3$$

The solution can be expressed analytically in the Fourier domain

$$\begin{aligned}
\hat{u}^h_1 &= \Big[-2B_1\beta^2 + \alpha\,\omega_1\,(B_1\omega_1 + B_2\,\omega_2)(1+\beta\,x_3) \\
&\qquad + \ \alpha\, i\omega_1\beta\, B_3(1-\alpha^{-1}+\beta x_3)\Big]\, e^{-\beta\,x_3} \\
\hat{u}^h_2 &= \Big[-2B_2\beta^2 + \alpha\,\omega_2\,(B_1\,\omega_1 + B_2\omega_2)(1+\beta\,x_3) \\
&\qquad + \ \alpha\, i\omega_2\beta\, B_3(1-\alpha^{-1}+\beta x_3)\Big]\, e^{-\beta\,x_3} \\
\hat{u}^h_3 &= \alpha\,\beta^2 \Big[i\,(\omega_1 B_1 + \omega_2 B_2)\,x_3 \\
&\qquad - \ B_3\,(\alpha^{-1}+\beta\,x_3)\,\Big]\, e^{-\beta\,x_3}
\end{aligned}$$

with:

$$B_1 = \frac{\hat{p}_1}{2\mu\,\beta^3}$$

$$B_2 = \frac{\hat{p}_2}{2\mu\,\beta^3}$$

$$B_3 = \frac{\beta\,\hat{p}_3 - i(1-\alpha)(\omega_1\hat{p}_1 + \omega_2\hat{p}_2)}{2\mu\,\alpha\,\beta^3(\beta+\gamma)}$$

Fourier-domain Green's function

A choice of a homogeneous term may satisfy the boundary condition.

The stress associated with a displacement field is

$$\sigma_{ij} = \mu \left(u_{i,j} + u_{j,i} - \frac{1-2\alpha}{1-\alpha} u_{k,k} \delta_{ij} \right)$$

or, in the Fourier domain

$$\hat{\sigma}_{ij} = \mu \left(k_j \delta_{il} + k_i \delta_{jl} - \frac{1-2\alpha}{1-\alpha} k_l \delta_{ij} \right) \hat{u}_l$$

At the surface, the traction is obtained by integration

recall

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(k)\, dk$$

$$\hat{t}_i^p(k_1, k_2) = \mu \int_{-\infty}^{\infty} \left(k_j \hat{u}_i^p + k_i \hat{u}_j^p - \frac{1-2\alpha}{1-\alpha} k_l \hat{u}_l^p \delta_{ij} \right) n_j \, dk_3$$

The homogeneous term must compensate

$$p_i = t_i^p + \Delta\rho \, g \, u_3^p n_i + q_i$$

Fourier-domain Green's function

Solving Navier's equation in the Fourier domain requires 4 steps:

$$\mu\left(\frac{\alpha}{1-\alpha}u_{j,ij}+u_{i,jj}\right)+f_i=0 \qquad \sigma_{ij}\hat{n}_j=q_j$$

1. Fourier transform the forcing term and apply the transfer function

$$\hat{u}_i^p=\frac{1}{\mu}\frac{(1-\alpha)\,k_lk_l\,\delta_{ij}-\alpha\,k_ik_j}{4\pi^2(k_lk_l)^2}\,\hat{f}_j$$

(N log N)

2. Evaluate the exceeding stress of the temporary solution in the Fourier domain

$$\hat{t}_i^p(k_1,k_2)=\mu\int_{-\infty}^{\infty}\left(k_j\hat{u}_i^p+k_i\hat{u}_j^p-\frac{1-2\alpha}{1-\alpha}\,k_l\hat{u}_l^p\delta_{ij}\right)n_j\,dk_3$$

(reduction N)

3. Compute and add the correction term

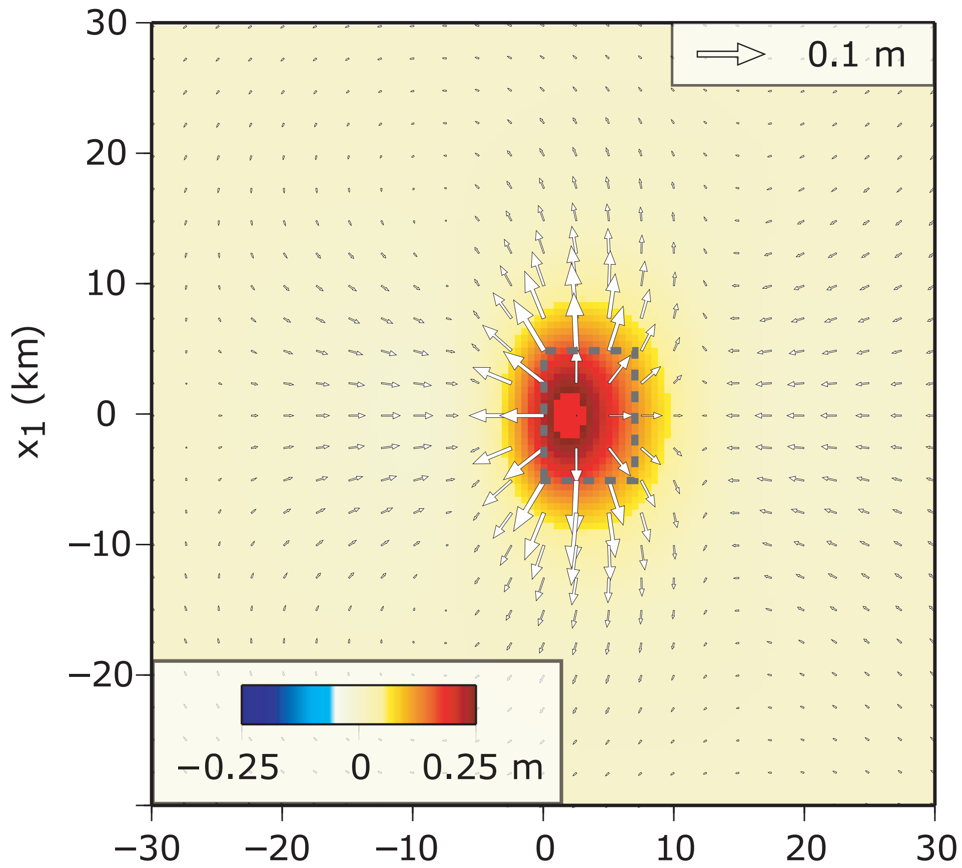
$$\hat{p}_i=\hat{t}_i^p+\Delta\rho\,g\,\hat{u}_3^pn_i+\hat{q}_i \qquad \hat{u}_i^h=\hat{u}_i^h(\hat{p}_i) \qquad u_i=u_i^h+u_i^p$$

(N)

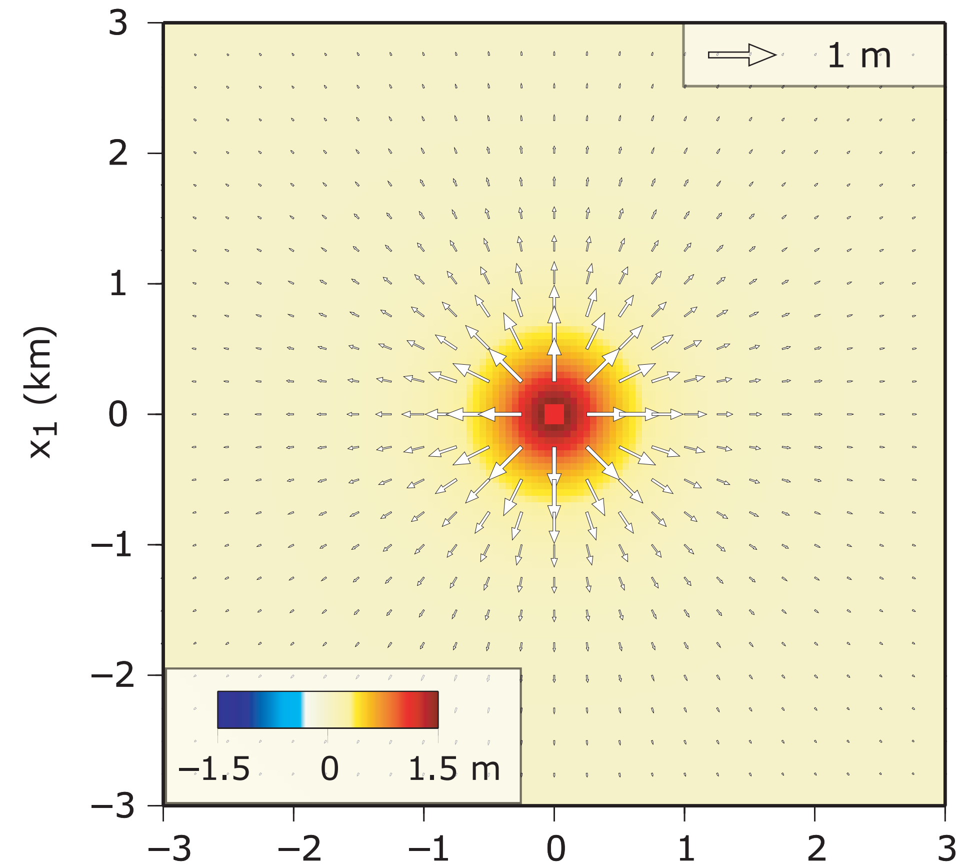
3. Inverse-Fourier transform **(N log N)**

Example calculation and benchmarks

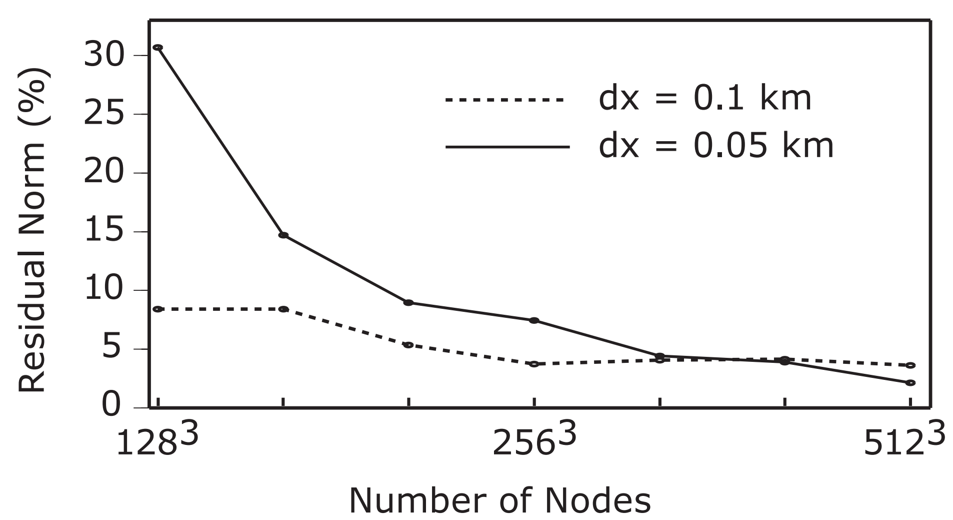
Thrust fault



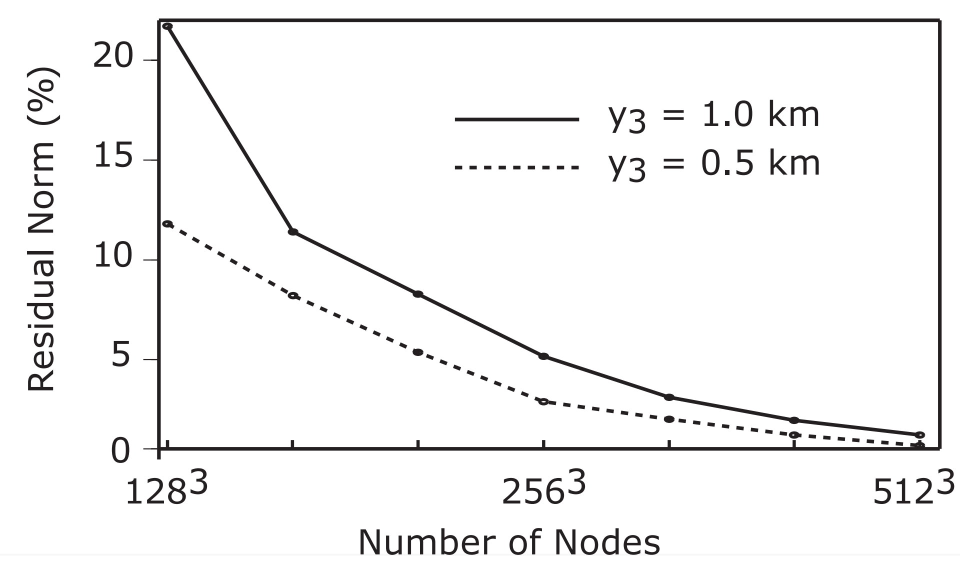
dilatation (Mogi) source



Numerical error



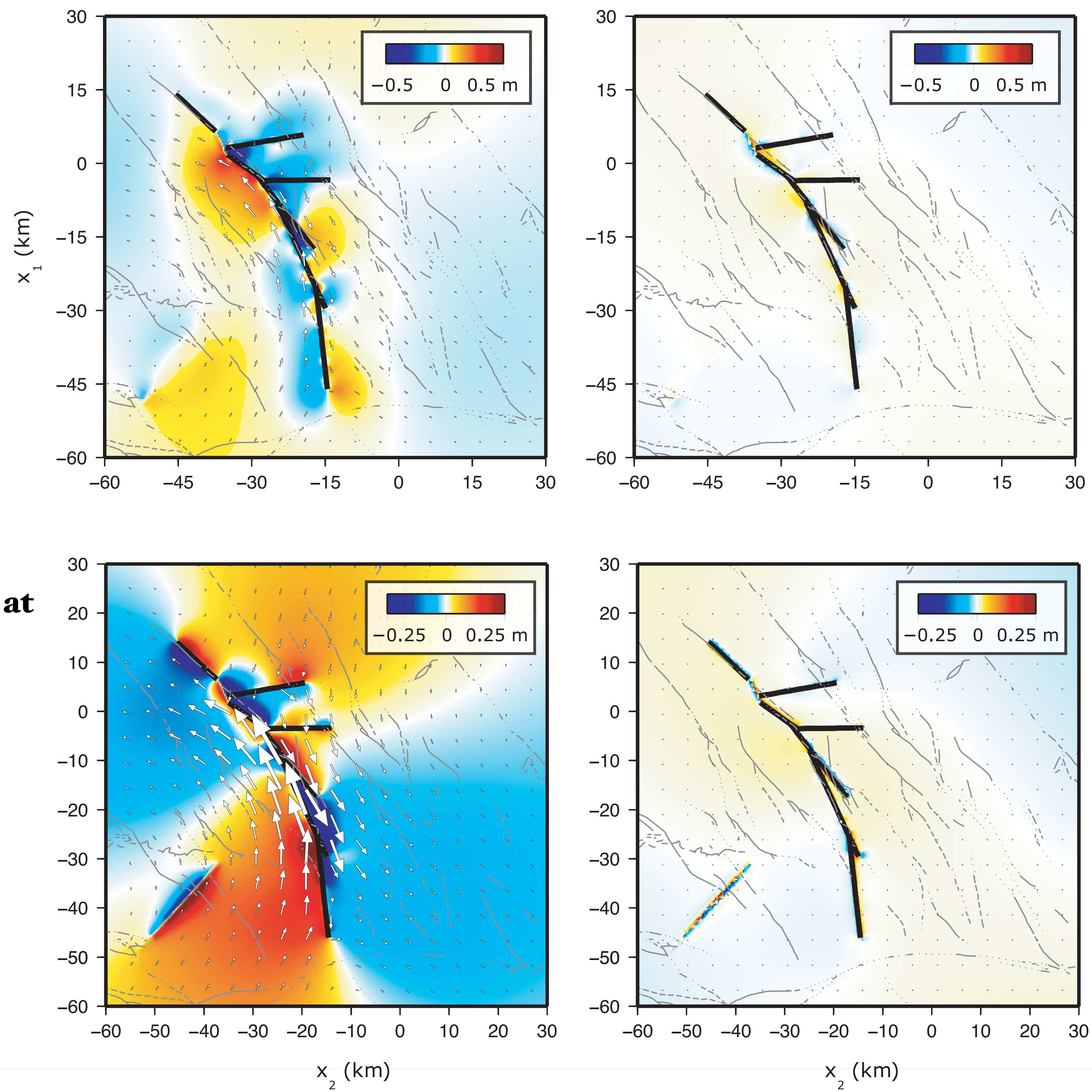
Numerical error



Case of the Mw 7.3 1992 Landers, CA earthquake

Surface displacements

residuals with the analytic formulation of Okada (1992)

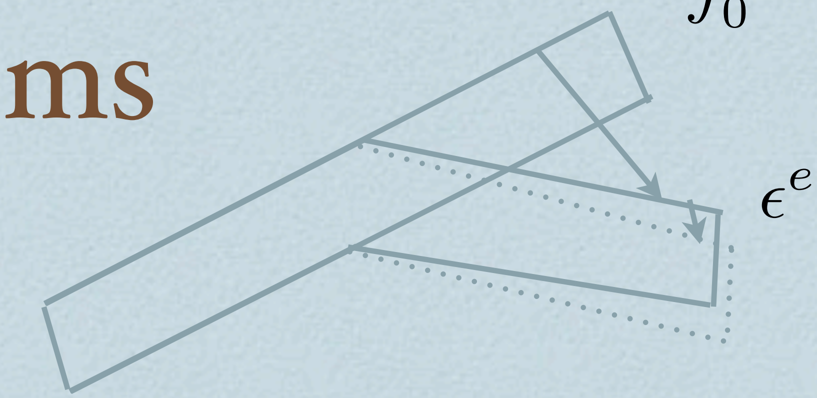


displacements at 10km depth

complex faulting geometry can be modeled using the superposition principle

Equivalent body forces for viscoelastoplastic problems

$$\epsilon_{ij}(t) = \int_0^t \dot{\epsilon}_{ij}\, dt$$



Total strain rate is decomposed in elastic and inelastic (eigenstrain) strain rate components

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^i_{ij}$$

Stress rate is the result of elastic (reversible) strain rate

$$\dot{\sigma}_{ij} = C_{ijkl}\left(\dot{\epsilon}_{kl} - \dot{\epsilon}^i_{kl}\right)$$

Conservation of momentum (Newton's law)

in the interior ⟶ $\dot{\sigma}_{ij}\, \hat{n}_j = 0$

$\dot{\sigma}_{ij,j} = 0$ ⟵ at the surface

inhomogeneous governing (Navier's) equation:

$$\left(C_{ijkl}\dot{\epsilon}_{kl}\right)_{,j} + \dot{f}_i = 0$$

boundary condition:

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equivalent body force (source term):

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equivalent surface traction:

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Equivalent body forces for viscoelastoplastic problems

Total strain rate is decomposed in elastic and inelastic (eigenstrain) strain rate components

$$\dot{\epsilon}_{ij} = \dot{\epsilon}^e_{ij} + \dot{\epsilon}^i_{ij}$$

The inelastic (irreversible) strain rate is decomposed into a rate and a strain direction

$$\dot{\epsilon}^i_{ij} = \dot{\gamma}\, R_{ij}$$

and the rate is controlled by a **constitutive law**

$$\dot{\gamma} = f(\sigma_{ij}, \gamma)$$

where the stress dependent on the current strain and the history of deformation **(hereditary equation)**

$$\sigma_{ij}(t) = C_{ijkl}\epsilon_{kl}(t) - \int_0^t \dot{\gamma}\, C_{ijkl} R_{kl}\, dt$$

Constitutive laws for relaxation problems

Poroelasticity:

$$R_{ij} = \frac{1}{3}\,\delta_{ij}$$

isotropic strain

Darcy flow

$$\dot\gamma = D\,\left[(1-\beta)\,\gamma - \beta\,\frac{\sigma}{\kappa_u}\right]_{,jj}$$

Fault creep:

$$R_{ij} = \frac{1}{2}\,(\Delta\hat\tau_i\hat n_j + \hat n_i\Delta\hat\tau_j)$$

dislocation

rate-and-state friction

$$\dot\gamma = 2\,\dot\gamma_0\,\sinh\frac{\Delta\tau}{(a-b)\bar\sigma}$$

Viscoelastic flow:

$$R_{ij} = \frac{\sigma'_{ij}}{\tau}$$

deviatoric strain

equation of state

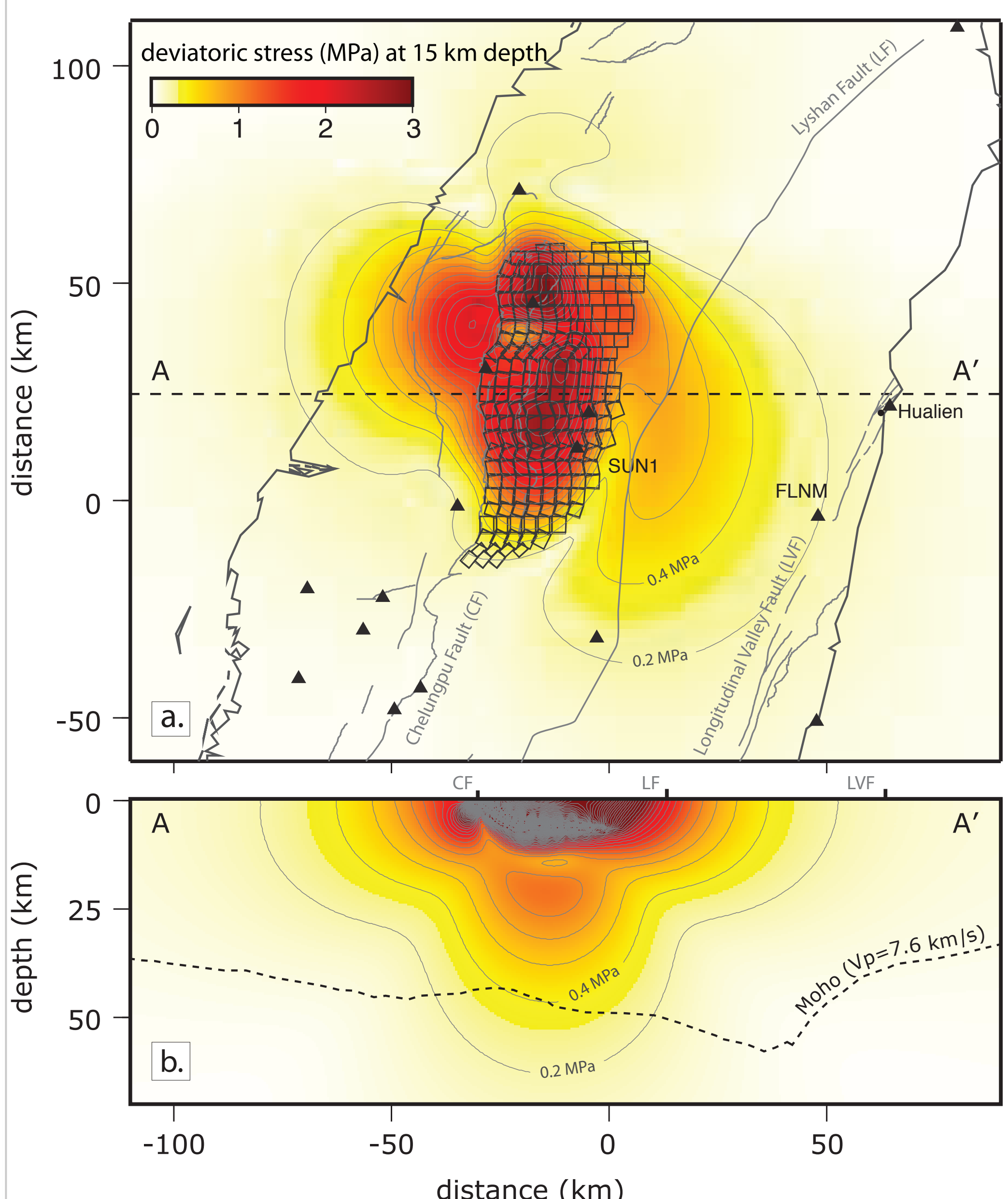
$$\dot\gamma = \dot\gamma_0\left(\frac{\tau}{G}\right)^n$$

Barbot & Fialko (2010b)

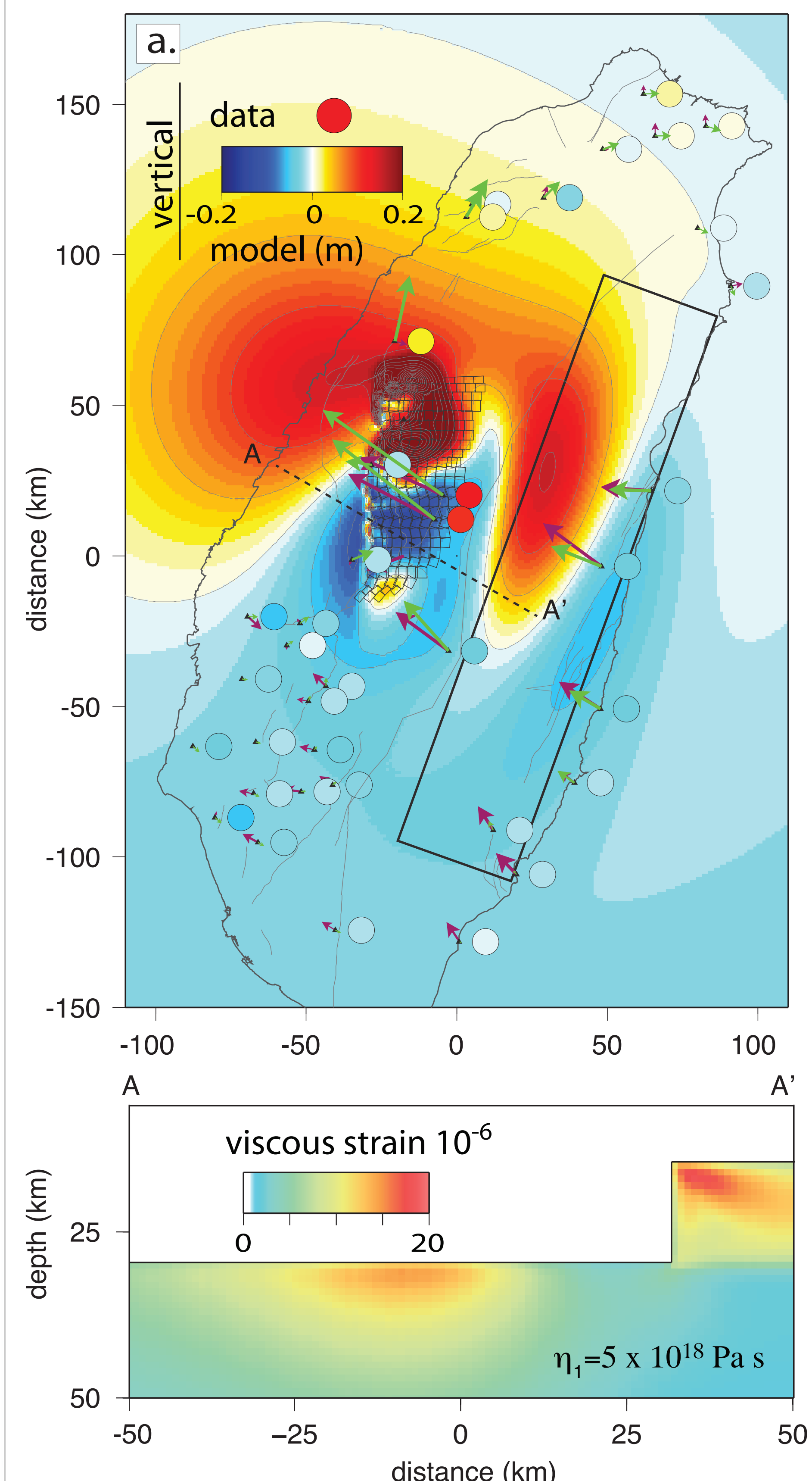
Relax Examples

Postseismic relaxation of the 1999 Mw 7.6 Chi-Chi earthquake

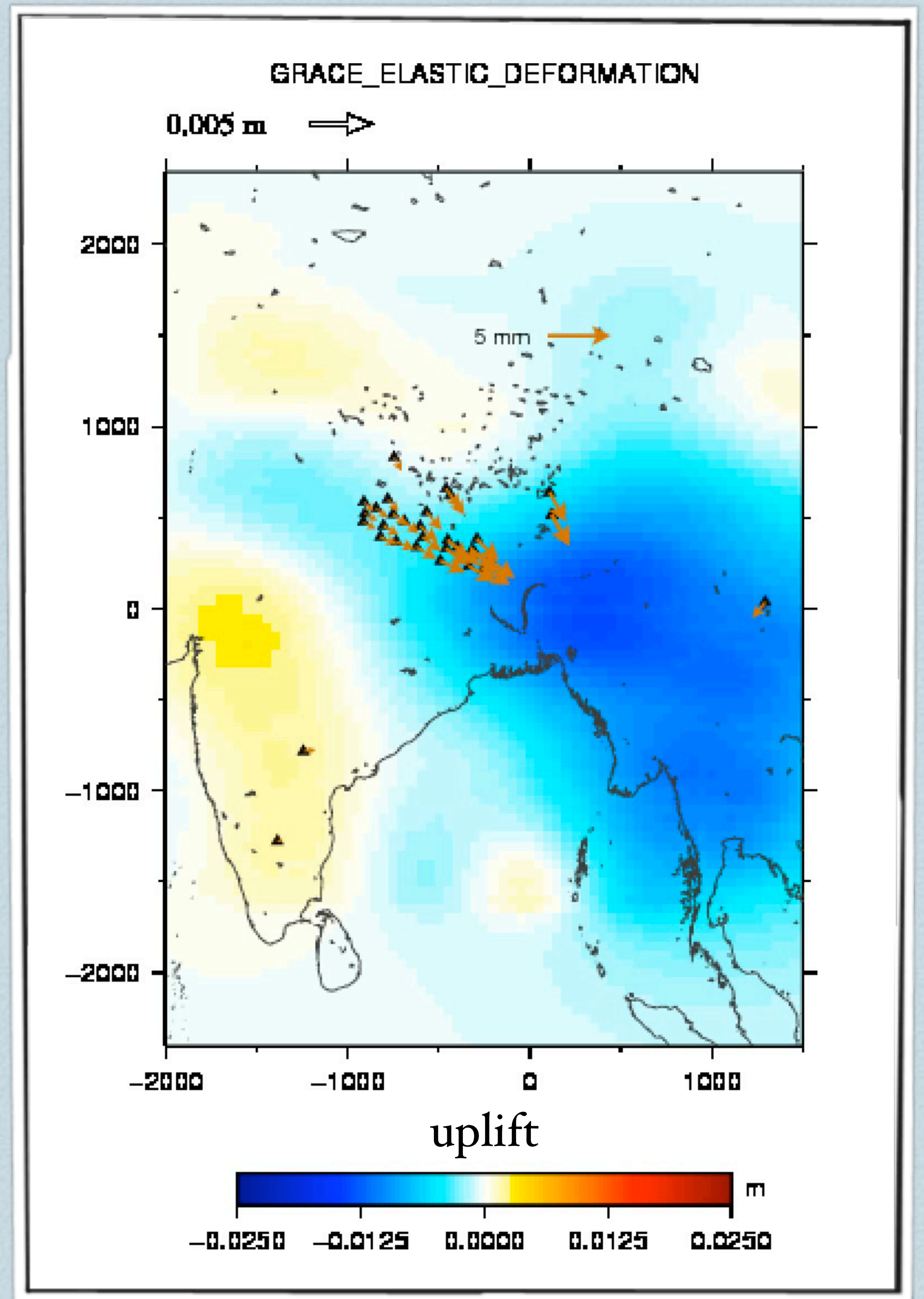
Coulomb stress calculation



Coupled afterslip and viscoelastic flow models



Rousset et al. (in prep.)



Surface processes

The change in surface loads from drainage of lakes (Cavalié et al. 2007), monsoons, or glaciers retreat (Larsen et al. 2005) can be use to constrain the rheological properties of the lithosphere.

Seasonal surface loads

Elastic or viscoelastic response to surface loads monitored by GRACE are modeled to compare with GPS time series.

Chanard et al. (in prep.)

Relax features

- Solves the elastic deformation in a homogeneous half space due to internal forces and surface tractions:
  - drainage of lakes, retreat of glaciers, monsoons,
  - earthquakes, magmatic inflation, dyke intrusions
- Solves the coupled nonlinear, fully heterogeneous viscoelastic deformation and fault creep due to stress perturbation:
  - postseismic deformation, volcanic unrest, afterslip,
  - regional postglacial rebound

Future improvements

- Lateral variations in elastic moduli (separate code today)
- Full effect of gravity, including the change in gravitational potential
- Approximation of the Earth's curvature
- Poroelasticity (separate code today)
- And many technical improvements (MPI, Cuda)